

The associated family of an elliptic surface and applications to minimal submanifolds

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Abstract

We show that elliptic surfaces in space forms for which all curvature ellipses of a certain order are circles allow a family of isometric deformations preserving the second fundamental form and the normal curvature tensor. For minimal surfaces this generates a one-parameter family of minimal isometric deformations that adds to the standard associated family. We also show how the associated family of a minimal Euclidean submanifold of rank two is determined by the associated family of an elliptic surface clarifying the geometry around the associated family of these higher dimensional submanifolds.

1 Introduction

It is a well-known fact that a simply connected minimal surface in a space form of any dimension allows a one-parameter family of isometric minimal deformations, called the associated family, and that in Euclidean space the family can be parametrically described with the classical Weierstrass representation; see [16]. A minimal surface is called m -isotropic if all ellipses of curvature up to order m are circles. The m -isotropic surfaces in Euclidean space are particularly distinguished since they can be constructed inductively by means of a Weierstrass type representation discussed below. As a special case of our general results for elliptic surfaces, we prove the following in the minimal case.

Theorem 1. *Any simply connected m -isotropic substantial surface $g: L^2 \rightarrow \mathbb{R}^N$ has an associated family of isometric m -isotropic surfaces $g_\theta: L^2 \rightarrow \mathbb{R}^N$ where $\theta \in \mathbb{S}^1 = [0, \pi)$. In addition, there exists a vector bundle isometry $\phi_\theta: N_g L \rightarrow N_{g_\theta} L$ that preserves the second fundamental form and the normal curvature tensor. Moreover, the associated family is trivial if and only if the surface g is holomorphic.*

That g is holomorphic means that there is a parallel complex structure in \mathbb{R}^N such that g is holomorphic in $C^{N/2} \equiv \mathbb{R}^N$. Including the usual associated family as a minimal

surface yields a two-parameter family but, of course, the second fundamental form is not preserved any longer.

In fact, the above result holds for m -isotropic surfaces in the sphere and hyperbolic space once holomorphicity is replaced by pseudoholomorphicity, which means that the ellipses of curvature of any order are circles and is equivalent to holomorphicity for Euclidean surfaces.

Euclidean submanifolds of rank two have been studied in different contexts; see [1], [7], [8], [11] and [12]. The submanifold having rank two means that the image of the Gauss map is a surface in the corresponding Grassmannian or, equivalently, that the kernel of the second fundamental form (relative nullity) has everywhere codimension two. The study of the minimal ones is particularly interesting since they belong to the important class of austere submanifolds introduced in [13]. Among them, we have the ones that carry a Kaehler structure described in [8] by means of a Weierstrass type representation in terms of m -isotropic surfaces.

Minimal submanifolds of rank two in the Euclidean space have been parametrically described in [8] by means of the class of elliptic surfaces for which the curvature ellipses of a certain order are all circles. A surface L^2 in a space form is called elliptic if its tangent bundle carries an almost complex structure J such that its normal valued second fundamental form satisfies

$$\alpha(X, X) + \alpha(JX, JX) = 0 \text{ for all } X \in TL.$$

It turns that the normal bundle of an elliptic surface splits as the orthogonal sum of a sequence of plane bundles (except the last one in odd codimension) such that each fiber contains an ellipse of curvature that is then ordered accordingly; see next section for details. Minimal surfaces can be seen as those elliptic surfaces for which the 0-ellipse of curvature is a circle, which is just a way to say that J is orthogonal.

To some surprise, it was observed in [9] that any simply connected minimal submanifold of rank two in Euclidean space of any dimension and codimension also allows an associated family of submanifolds of the same class. As in the surface case, this family is obtained by rotating the second fundamental form while keeping fixed the normal bundle and the induced normal connection. This fact together with the representation in [8] discussed above, suggests that an elliptic surface in a space form for which all ellipses of curvature of a certain order are circles should have some kind of associate family preserving that property, and that was the starting point of this paper. Here, we answer the question in the affirmative showing the existence of an associated family depending on one parameter.

There is an abundance of examples of surfaces with circular ellipses of curvature, specially minimal ones. In particular, there are the surfaces for which all but the last one ellipse of curvature is a circle. These have been studied in the sphere [3] and in hyperbolic space [17] under the name of superconformal. Other interesting examples are Lawson's surfaces and the holomorphic curves in the nearly Kaehler sphere S^6 .

Most of what is done in this paper for surfaces can be extended to elliptic submanifolds of rank two. But in the final section of the paper, we limit ourselves to show how the associated family of a minimal Euclidean submanifold of rank two is determined by the associated family to an elliptic surface with a circular ellipse of curvature. This result completely clarifies the geometry around the associated family of these higher dimensional submanifolds.

Finally, we observe that a key ingredient of our proofs is the classical Burstin-Mayer-Allendoerfer theory as discussed in Vol. IV of Spivak [19]. Similar to the case of curves, this theory shows that certain tensors associated to a set of Frenet type equations are a complete set of invariants for a submanifold of a space form. Among these tensors, one has the higher order fundamental forms some of which are preserved in our case. We should point out that isometric deformations of submanifolds that also preserve higher fundamental forms, starting with the second fundamental form, up to a stated order was already considered in [4].

2 Preliminaries

In this section, we first recall from [19] some basic definitions for submanifolds in space forms. Then, we recall from [8] the notions of elliptic surface, ellipse of curvature and polar surface to an elliptic surface and discuss some of their basic properties.

Let $f: M^n \rightarrow \mathbb{Q}_c^N$ be a substantial isometric immersion of a connected n -dimensional Riemannian manifold into either the Euclidean space \mathbb{R}^N ($c = 0$), the Euclidean sphere \mathbb{S}^N ($c > 0$) or the hyperbolic space \mathbb{H}^N ($c < 0$) with vector valued second fundamental form α_f and induced connection ∇^\perp in the normal bundle $N_f M$. Being *substantial* (or full) means that the codimension cannot be reduced.

The k^{th} -normal space $N_k^f(x)$ of f at $x \in M^n$ for $k \geq 1$ is defined as

$$N_k^f(x) = \text{span}\{\alpha_f^{k+1}(X_1, \dots, X_{k+1}) : X_1, \dots, X_{k+1} \in T_x M\}.$$

Here $\alpha_f^2 = \alpha_f$ and for $s \geq 3$ the symmetric tensor $\alpha_f^s: TM \times \dots \times TM \rightarrow N_f M$ called the s^{th} -fundamental form is defined inductively by

$$\alpha_f^s(X_1, \dots, X_s) = (\nabla_{X_s}^\perp \dots \nabla_{X_3}^\perp \alpha_f(X_2, X_1))^\perp$$

where $(\)^\perp$ denotes taking the projection onto the normal subspace $(N_1^f \oplus \dots \oplus N_{s-1}^f)^\perp$.

We always admit that the immersion f is *regular* (called nicely curved in [19]) which means that all N_k^f 's have constant dimension for each k and therefore form normal subbundles. Geometrically, this means that at each point the submanifold bends in the same number of directions. Notice that for any submanifold this condition is verified along connected components of an open dense subset of M^n .

A surface $g: L^2 \rightarrow \mathbb{Q}_c^N$ was called *elliptic* in [8] if there exists a (unique up to a sign) almost complex structure $J: TL \rightarrow TL$ such that the second fundamental form satisfies

$$\alpha_g(X, X) + \alpha_g(JX, JX) = 0 \text{ for all } X \in TL.$$

For a regular elliptic surface all normal spaces have dimension two except the last one that is one-dimensional if N is odd. Thus, the normal bundle $N_g L$ splits as

$$N_g L = N_1^g \oplus \cdots \oplus N_{\tau_g}^g,$$

where τ_g (sometimes called the geometric degree of g) is the index of the last subbundle.

Setting

$$\tau_g^o = \begin{cases} \tau_g & \text{if } N \text{ is even} \\ \tau_g - 1 & \text{if } N \text{ is odd,} \end{cases}$$

it turns out that the almost complex structure $J = J_0$ on TL induces an almost complex structure J_s on each N_s^g , $1 \leq s \leq \tau_g^o$, defined by

$$J_s \alpha_g^{s+1}(X_1, \dots, X_{s+1}) = \alpha_g^{s+1}(JX_1, \dots, X_{s+1}). \quad (1)$$

In the sequel, we denote by $\pi_s: N_g L \rightarrow N_s^g$, $1 \leq s \leq \tau_g$, the orthogonal projection. We have for $2 \leq s \leq \tau_g^o$ from [8, Prop. 5] that

$$J_s \pi_s(\nabla_X^\perp \xi) = \pi_s(\nabla_X^\perp J_{s-1} \xi) = \pi_s(\nabla_{JX}^\perp \xi) \text{ if } \xi \in N_{s-1}^g \quad (2)$$

and

$$J_{s-1}^t \pi_{s-1}(\nabla_X^\perp \xi) = \pi_{s-1}(\nabla_X^\perp J_s^t \xi) = \pi_{s-1}(\nabla_{JX}^\perp \xi) \text{ if } \xi \in N_s^g. \quad (3)$$

Take any $\varphi \in \mathbb{S}^1 = [0, \pi)$ and let $R_\varphi^s: N_s^g \rightarrow N_s^g$, $1 \leq s \leq \tau_g^o$, denote the map given by

$$R_\varphi^s = \cos \varphi I + \sin \varphi J_s. \quad (4)$$

It follows from (2) and (3) that

$$R_\varphi^{s+1} \pi_{s+1}(\nabla_X^\perp \xi) = \pi_{s+1}(\nabla_X^\perp R_\varphi^s \xi) \text{ if } \xi \in N_s^g \quad (5)$$

and

$$(R_\varphi^s)^t \pi_s(\nabla_X^\perp \xi) = \pi_s(\nabla_X^\perp (R_\varphi^{s+1})^t \xi) \text{ if } \xi \in N_{s+1}^g \quad (6)$$

for any $1 \leq s \leq \tau_g^o - 1$.

The s^{th} -order curvature ellipse $\mathcal{E}_s^g(x) \subset N_s^g(x)$, $1 \leq s \leq \tau_g^o$, of g at $x \in L^2$ is

$$\mathcal{E}_s^g(x) = \{\alpha_g^{s+1}(Z_\psi, \dots, Z_\psi) : Z_\psi = \cos \psi Z + \sin \psi JZ \text{ and } \psi \in [0, \pi)\},$$

where $Z \in T_x L$ has unit length and satisfies $\langle Z, JZ \rangle = 0$. It follows from the ellipticity condition that such a Z always exists and that $\mathcal{E}_s^g(x)$ is indeed an ellipse.

We point out that in the case of the first ellipse of curvature the above definition coincides with the standard definition only if the mean curvature vanishes, in which case the higher order ellipses also coincide.

For this paper, a fundamental fact is that $\mathcal{E}_s^g(x)$ is a circle if and only if $J_s(x)$ is orthogonal. Throughout the paper by \mathcal{E}_ℓ^g being a circle we mean that the curvature ellipse $\mathcal{E}_\ell^g(x)$ is a circle for any $x \in L^2$.

A polar surface to an elliptic surface $g: L^2 \rightarrow \mathbb{Q}_c^{N-c} \subset \mathbb{R}^N$ ($c = 0, 1$) is an immersion defined as follows:

- (i) If $N - c$ is odd, then the polar surface $h: L^2 \rightarrow \mathbb{S}_1^{N-1}$ is the spherical image of a unit normal field spanning the last one-dimensional normal bundle.
- (ii) If $N - c$ is even, then the polar surface $h: L^2 \rightarrow \mathbb{R}^N$ is any surface such that $T_{h(x)}L = N_{\tau_g}^g(x)$ up to parallel identification in \mathbb{R}^N .

It is known that any elliptic surface in case (ii) admits locally many polar surfaces.

It turns out that a polar surface to an elliptic surface is necessarily elliptic. Moreover, if the elliptic surface has a circular ellipse of curvature then its polar surface has the same property at the “corresponding” normal bundle; see [8] for details. In particular, for the polar surface to an m -isotropic surface the last $m + 1$ ellipses of curvature are circles. Notice that in this case the polar surface is not necessarily minimal.

3 Examples

In this section, we discuss several families of examples of surfaces that carry circular ellipses of curvature that have been considered in the literature.

The m -isotropic surfaces: A minimal surface in a space form is called m -isotropic when all the ellipses of curvature up to order m are circles. It is well known that holomorphic curves in \mathbb{C}^p are precisely the $(p - 1)$ -isotropic surfaces in \mathbb{R}^{2p} .

The following parametric description of the m -isotropic surfaces in Euclidean space was given in [10] based on results in [6]. On a simply connected domain $U \subset \mathbb{C}$, a minimal surface $h: U \rightarrow \mathbb{R}^N$ has the Weierstrass representation $h = \operatorname{Re} \int^z \gamma dz$ where the Gauss map $\gamma: U \rightarrow \mathbb{C}^N$ of h is given by

$$\gamma = (\beta/2) (1 - \phi^2, i(1 + \phi^2), 2\phi),$$

being β holomorphic and $\phi: U \rightarrow \mathbb{C}^{N-2}$ meromorphic; see [16] for details. We have that h is m -isotropic if and only if $(\phi', \phi') = \dots = (\phi^{(m)}, \phi^{(m)}) = 0$, where $(,)$ stands for the standard symmetric inner product in \mathbb{C}^{N-2} . Therefore, to construct any m -isotropic

surface we start with a nonzero holomorphic map $\alpha_0: U \rightarrow \mathbb{C}^{N-2(m+1)}$. Assuming that $\alpha_r: U \rightarrow \mathbb{C}^{N-2(m-r+1)}$, $0 \leq r \leq m$, has been defined already, set

$$\alpha_{r+1} = \beta_{r+1} (1 - \phi_r^2, i(1 + \phi_r^2), 2\phi_r)$$

where $\phi_r = \int^z \alpha_r dz$ and $\beta_{r+1} \neq 0$ is any holomorphic function. Then, the elliptic surface $g = \text{Re } \phi_{m+1}$ in \mathbb{R}^N is m -isotropic and has Gauss map $\gamma = \alpha_{m+1}$.

Lawson's surfaces: These are minimal surfaces in spheres that decompose as a direct sum of elements in the associated family h_θ , $0 \leq \theta < \pi$, of a minimal surface h in \mathbb{S}^3 . More precisely, we consider surfaces in $\mathbb{S}^{4n-1} \subset \mathbb{R}^{4n}$ given as

$$f = a_1 h_{\theta_1} \oplus \dots \oplus a_n h_{\theta_n},$$

where $0 \leq \theta_1 < \dots < \theta_n < \pi$, the real numbers a_1, \dots, a_n satisfy $\sum_{j=1}^n a_j^2 = 1$ and \oplus denotes the orthogonal sum with respect to an orthogonal decomposition of \mathbb{R}^{4n} . It has been checked in [20] that all ellipses of curvature of even order are circles and that the ones of odd order generically are not. These surfaces are related to Lawson's conjecture [18] which asserts that the only non-flat minimal surfaces in spheres that are locally isometric to minimal surfaces in \mathbb{S}^3 are Lawson's surfaces.

Holomorphic curves in the nearly Kaehler sphere \mathbb{S}^6 : It is well known that the multiplicative structure on the Cayley numbers can be used to define a non-integrable almost complex structure J on the sphere \mathbb{S}^6 that is nearly Kaehler. A holomorphic curve in \mathbb{S}^6 is a non-constant map g from a Riemann surface into the nearly Kaehler sphere \mathbb{S}^6 whose differential is complex linear. It turns out that its second fundamental form satisfies

$$\alpha_g(JX, Y) = J\alpha_g(X, Y).$$

It follows that g is minimal and its first curvature ellipse is a circle. There is a family of these surfaces for which the second curvature ellipse is not a circle. The theory of these holomorphic curves was started in [5] and developed in [2], [14] and [15].

4 The results

In this section, we state our results on the associated family to an elliptic surface with circular ellipses of curvature. We first assert the existence of the associated family and then discuss when the family is trivial.

Theorem 2. *Let $g: L^2 \rightarrow \mathbb{Q}_c^N$, $N \geq 6$, be a simply connected substantial elliptic surface with \mathcal{E}_ℓ^g a circle for some $1 \leq \ell \leq \tau_g^o - 1$. For each $\theta \in \mathbb{S}^1$ there exists an elliptic isometric immersion $g_\theta: L^2 \rightarrow \mathbb{Q}_c^N$ with respect to the same almost complex structure and a vector bundle isometry $\phi_\theta: N_g L \rightarrow N_{g_\theta} L$ that preserves the second fundamental form and the normal curvature tensor.*

For $\ell \geq 2$ it turns out that also some of the other higher fundamental forms are preserved, i.e., $\alpha_{g_\theta}^k = \phi_\theta \circ \alpha_g^k$. In fact, it follows from Proposition 8 below that this fact holds up to order $k \leq \ell + 1$. It also follows from that result that the property of an ellipse of curvature being a circle remains true for the associated family.

Definition 3. We call the *associated family* to an elliptic surface $g: L^2 \rightarrow \mathbb{Q}_c^N$ with \mathcal{E}_ℓ^g a circle for some $1 \leq \ell \leq \tau_g^o - 1$ the set

$$G_\ell = \{g_\theta: L^2 \rightarrow \mathbb{Q}_c^N : \theta \in \mathbb{S}^1 = [0, \pi)\}$$

of elliptic isometric immersions given by Theorem 2.

By the associated family being *trivial* we mean that G_ℓ only contains one element, i.e., any g_θ is congruent to g in the ambient space.

Theorem 4. Let $g: L^2 \rightarrow \mathbb{Q}_c^N$, $N \geq 6$, be a simply connected substantial elliptic surface with \mathcal{E}_ℓ^g a circle for some $1 \leq \ell \leq \tau_g^o - 1$. If the surfaces $g_\theta, g_{\tilde{\theta}} \in G_\ell$ are congruent for $\theta \neq \tilde{\theta}$, then the \mathcal{E}_s^g are circles for all $\ell \leq s \leq \tau_g^o$. Conversely, the associated family G_ℓ is trivial if the \mathcal{E}_s^g are circles for $\ell \leq s \leq \tau_g^o$.

The following is a consequence of the above result and basic properties of polar surfaces of elliptic surfaces.

Corollary 5. Let $g: L^2 \rightarrow \mathbb{Q}_c^N$, $c = 0, 1$, be a simply connected substantial elliptic surface with \mathcal{E}_ℓ^g a circle for some $1 \leq \ell \leq \tau_g^o - 1$. Then the associated family G_ℓ is trivial if and only if g is (locally) a polar surface to an m -isotropic surface for $m = \tau_g^o - \ell$.

5 The proofs

5.1 The compatibility equations

A key ingredient in the proofs are the basic equations from the classical Burstin-Mayer-Allendoerfer theory discussed in Vol. IV of [19]. They naturally extend the situation for curves under similar regularity conditions. The main result is that for a submanifold of a space form the tensors determined by the Frenet equations are a complete set of invariants.

The Frenet equations for a regular isometric immersion $f: M^n \rightarrow \mathbb{Q}_c^N$ are given by

$$\tilde{\nabla}_X \xi = -A_\xi^s X + D_X^s \xi + S_X^s \xi \quad \text{if } \xi \in N_s^f \text{ and } X \in T_x M, \quad s \geq 1,$$

in terms of the linear maps

$$A^s: TM \times N_s^f \rightarrow N_{s-1}^f \quad \text{defined by} \quad A_\xi^s X = -\pi_{s-1}(\tilde{\nabla}_X \xi),$$

$$\begin{aligned}
D^s: TM \times N_s^f &\rightarrow N_s^f & \text{defined by } D_X^s \xi &= \pi_s(\nabla_X^\perp \xi), \\
S^s: TM \times N_s^f &\rightarrow N_{s+1}^f & \text{defined by } S_X^s \xi &= \pi_{s+1}(\nabla_X^\perp \xi),
\end{aligned}$$

where $\tilde{\nabla}$ is the connection in the induced bundle $f^*(T\mathbb{Q}_c^N) = N_0^f \oplus N_f M$ and π_0 is the projection onto $N_0^f = f_*(TM)$. Notice that A_ξ^1 is the standard Weingarten operator and that D^s is a connection in N_s^f compatible with the metric. An important fact is that the tensors A^s and S^s are completely determined by the higher fundamental forms since

$$S_X^s(\alpha_f^{s+1}(X_1, \dots, X_{s+1})) = \alpha_f^{s+2}(X, X_1, \dots, X_{s+1})$$

and

$$\langle A_\xi^s X, \eta \rangle = \langle \xi, S_X^{s-1} \eta \rangle \text{ for } \xi \in N_s^f \text{ and } \eta \in N_{s-1}^f. \quad (7)$$

We briefly summarize the basic results of the theory: Let $f, \tilde{f}: M^n \rightarrow \mathbb{Q}_c^N$ be two regular isometric immersions. If there are vector bundle isometries $\phi_k: N_k^f \rightarrow N_k^{\tilde{f}}$ for all $k \geq 1$, which preserve the fundamental forms α^{k+1} and the induced normal connections D^k , then there is an isometry τ of \mathbb{Q}_c^N such that $\tilde{f} = \tau \circ f$ and $\phi_k = \tau_*|_{N_k^f}$. Moreover, there is a set of equations given below, namely, the Generalized Gauss and Codazzi equations, that relate the higher fundamental forms and the induced connections. It turns out that the set of connections D^k in N_k^f is the unique set for which the higher order fundamental forms satisfy the Codazzi equations. Furthermore, the Generalized Gauss and Codazzi equations are the integrability conditions that assure the existence of an isometric immersion provided all data involved has been provided.

The Generalized Gauss equation.

$$A_{S_Y^s \xi}^{s+1} X - A_{S_X^s \xi}^{s+1} Y = D_X^s D_Y^s \xi - D_Y^s D_X^s \xi - S_X^{s-1} A_\xi^s Y + S_Y^{s-1} A_\xi^s X - D_{[X, Y]}^s \xi \quad (8)$$

for all $X, Y \in TM$ and $\xi \in N_s^f$.

The Generalized Codazzi equation.

$$D_X^{s+1}(S_Y^s \xi) - D_Y^{s+1}(S_X^s \xi) + S_X^s D_Y^s \xi - S_Y^s D_X^s \xi - S_{[X, Y]}^s \xi = 0 \quad (9)$$

for all $X, Y \in TM$ and $\xi \in N_s^f$.

Using (7) we have that (9) has the equivalent form

$$D_X^s A_\xi^{s+1} Y - D_Y^s A_\xi^{s+1} X + A_{D_Y^{s+1} \xi}^{s+1} X - A_{D_X^{s+1} \xi}^{s+1} Y - A_\xi^{s+1}[X, Y] = 0 \quad (10)$$

for all $X, Y \in TM$ and $\xi \in N_{s+1}^f$.

We conclude with some useful symmetric equations.

Proposition 6. *It holds that*

$$\mathbf{S}_Y^{s+1}\mathbf{S}_X^s\xi = \mathbf{S}_X^{s+1}\mathbf{S}_Y^s\xi \text{ or, equivalently, that } A_{A_\xi^s X}^{s-1}Y = A_{A_\xi^s Y}^{s-1}X \quad (11)$$

for any $\xi \in N_s^f$ and $X, Y \in TM$.

Proof: To prove the first equation take $\xi = \alpha_f^{s+1}(X_1, \dots, X_{s+1})$ and use the symmetry of the higher fundamental forms. For the proof of the equivalent second equation take $\xi \in N_s^f$, $\eta \in N_{s-2}^f$ and use (7) twice to obtain

$$\langle A_{A_\xi^s X}^{s-1}Y - A_{A_\xi^s Y}^{s-1}X, \eta \rangle = \langle \xi, \mathbf{S}_X^{s-1}\mathbf{S}_Y^{s-2}\eta - \mathbf{S}_Y^{s-1}\mathbf{S}_X^{s-2}\eta \rangle = 0,$$

and this concludes the proof. ■

5.2 The proofs

For a substantial elliptic surface $g: L^2 \rightarrow \mathbb{Q}_c^N$ with a circular ellipse of curvature in a space form we first define a one-parameter family of compatible connections in the normal bundle. Hereafter, we assume that \mathcal{E}_ℓ^g is a circle for given $1 \leq \ell \leq \tau_g^o - 1$, i.e., the almost complex structure J_ℓ is a vector bundle isometry. Notice that J_ℓ is parallel with respect to the induced connection on N_ℓ^g by dimension reasons. Thus, for any $\varphi \in \mathbb{S}^1 = [0, \pi)$ the map $R_\varphi^\ell: N_\ell^g \rightarrow N_\ell^g$ defined by (4) is also a parallel isometry, i.e.,

$$\pi_\ell(\nabla_X^\perp R_\varphi^\ell \xi) = R_\varphi^\ell \pi_\ell(\nabla_X^\perp \xi). \quad (12)$$

For each $\theta \in \mathbb{S}^1$, consider the map $\nabla^\theta: TL \times N_g L \rightarrow N_g L$ defined modifying the normal connection of g as follows:

$$\begin{cases} \pi_{\ell+1}(\nabla_X^\theta \xi) = \pi_{\ell+1}(\nabla_X^\perp R_\theta^\ell \xi) & \text{if } \xi \in N_\ell^g \\ \pi_\ell(\nabla_X^\theta \eta) = R_{-\theta}^\ell \pi_\ell(\nabla_X^\perp \eta) & \text{if } \eta \in N_{\ell+1}^g, \end{cases} \quad (13)$$

and $\nabla^\theta = \nabla^\perp$ in all other cases.

Lemma 7. *The map ∇^θ is a Riemannian connection with curvature tensor $R^\theta = R^\perp$.*

Proof: Take $\xi \in N_\ell^g$ and $\eta \in N_{\ell+1}^g$. Then,

$$\nabla_X^\theta f\xi = (\pi_{\ell-1} + \pi_\ell)(\nabla_X^\perp f\xi) + \pi_{\ell+1}(\nabla_X^\perp fR_\theta^\ell \xi) = X(f)\xi + f\nabla_X^\theta \xi \quad (14)$$

and

$$\nabla_X^\theta f\eta = (\pi_{\ell+1} + \pi_{\ell+2})(\nabla_X^\perp f\eta) + R_{-\theta}^\ell \pi_\ell(\nabla_X^\perp f\eta) = X(f)\eta + f\nabla_X^\theta \eta. \quad (15)$$

Moreover, we obtain using (12) that

$$\langle \nabla_X^\theta \xi, \eta \rangle + \langle \xi, \nabla_X^\theta \eta \rangle = \langle \nabla_X^\perp R_\theta^\ell \xi, \eta \rangle + \langle \xi, R_{-\theta}^\ell \nabla_X^\perp \eta \rangle = \langle R_\theta^\ell \nabla_X^\perp \xi, \eta \rangle + \langle \xi, R_{-\theta}^\ell \nabla_X^\perp \eta \rangle = 0, \quad (16)$$

and that the connection is Riemannian follows easily from (14), (15) and (16).

The second claim amounts to show that the tensor defined by

$$\mathcal{R}(X, Y)\xi = R^\theta(X, Y)\xi - R^\perp(X, Y)\xi$$

vanishes. In the sequel, to avoid writing rather long but straightforward computations some of the arguments will just be sketched.

We divide the proof in several cases:

Case 1. The case $\xi \in N_1^g \oplus \cdots \oplus N_{\ell-2}^g \oplus N_{\ell+3}^g \oplus \cdots \oplus N_{\tau_g}^g$ is trivial.

Case 2. Take $\xi \in N_{\ell+2}^g$. Then,

$$\mathcal{R}(X, Y)\xi = (R_{-\theta}^\ell - I) \left(A_{A_\xi^{\ell+2}Y}^{\ell+1} X - A_{A_\xi^{\ell+2}X}^{\ell+1} Y \right),$$

and the claim follows from (11).

Case 3. Take $\xi \in N_{\ell+1}^g$. Then

$$\mathcal{R}(X, Y)\xi = B_\theta(X, Y) - B_\theta(Y, X) - B_0(X, Y) + B_0(Y, X) + (I - R_{-\theta}^\ell) \pi_\ell(\nabla_{[X, Y]}^\perp \xi)$$

where

$$B_\theta(X, Y) = R_{-\theta}^\ell \pi_\ell(\nabla_X^\perp \pi_{\ell+1}(\nabla_Y^\perp \xi)) + \pi_\ell(\nabla_X^\perp R_{-\theta}^\ell \pi_\ell(\nabla_Y^\perp \xi)) + \pi_{\ell-1}(\nabla_X^\perp R_{-\theta}^\ell \pi_\ell(\nabla_Y^\perp \xi)).$$

Observe that (12) can be written as

$$D_X^\ell R_\varphi^\ell \xi = R_\varphi^\ell D_X^\ell \xi. \quad (17)$$

We obtain using (11) and (17) that

$$\begin{aligned} \mathcal{R}(X, Y)\xi &= (I - R_{-\theta}^\ell) \left(D_X^\ell A_\xi^{\ell+1} Y - D_Y^\ell A_\xi^{\ell+1} X + A_{D_Y^{\ell+1} \xi}^{\ell+1} X - A_{D_X^{\ell+1} \xi}^{\ell+1} Y - A_\xi^{\ell+1} [X, Y] \right) \\ &\quad + A_{R_{-\theta}^\ell A_\xi^{\ell+1} Y}^\ell X - A_{R_{-\theta}^\ell A_\xi^{\ell+1} X}^\ell Y. \end{aligned}$$

For $\eta \in N_{\ell-1}^g$, we have using (7) that

$$\langle A_{R_{-\theta}^\ell A_\xi^{\ell+1} Y}^\ell X, \eta \rangle = \langle A_\xi^{\ell+1} Y, R_\theta^\ell S_X^{\ell-1} \eta \rangle = \langle \xi, S_Y^\ell R_\theta^\ell S_X^{\ell-1} \eta \rangle. \quad (18)$$

Since $R_\theta^\ell S_X^{\ell-1} \xi = S_X^{\ell-1} R_\theta^{\ell-1} \xi$ from (5), we obtain from (11) that

$$S_Y^\ell R_\theta^\ell S_X^{\ell-1} \xi = S_X^\ell R_\theta^\ell S_Y^{\ell-1} \xi. \quad (19)$$

Now the claim follows from (10), (18) and (19).

Case 4. Take $\xi \in N_\ell^g$. First assume $\ell \geq 2$. Using (8), (9), (11) and (17) we obtain

$$R^\theta(X, Y)\xi = S_Y^{\ell-1}A_\xi^\ell X - S_X^{\ell-1}A_\xi^\ell Y - R_{-\theta}^\ell(S_Y^{\ell-1}A_{R_\theta^\ell \xi}^\ell X - S_X^{\ell-1}A_{R_\theta^\ell \xi}^\ell Y). \quad (20)$$

On the other hand, it holds that

$$R_\varphi^\ell(S_Y^{\ell-1}A_\xi^\ell X - S_X^{\ell-1}A_\xi^\ell Y) = S_Y^{\ell-1}A_{R_\varphi^\ell \xi}^\ell X - S_X^{\ell-1}A_{R_\varphi^\ell \xi}^\ell Y. \quad (21)$$

In fact, it follows using (7) that

$$\langle R_\varphi^\ell(S_Y^{\ell-1}A_\xi^\ell X - S_X^{\ell-1}A_\xi^\ell Y), R_\varphi^\ell \delta \rangle = \langle A_\delta^\ell Y, A_\xi^\ell X \rangle - \langle A_\delta^\ell X, A_\xi^\ell Y \rangle$$

and

$$\langle S_Y^{\ell-1}A_{R_\varphi^\ell \xi}^\ell X - S_X^{\ell-1}A_{R_\varphi^\ell \xi}^\ell Y, R_\varphi^\ell \delta \rangle = \langle A_{R_\varphi^\ell \delta}^\ell Y, A_{R_\varphi^\ell \xi}^\ell X \rangle - \langle A_{R_\varphi^\ell \delta}^\ell X, A_{R_\varphi^\ell \xi}^\ell Y \rangle.$$

Since $N_\ell^g = \text{span}\{\xi, J_\ell \xi\}$, to obtain (21) is suffices to compute the right hand side of both equations for $\delta = J_\ell \xi$ and observe that they coincide.

We now have from (20) and (21) that $R^\theta(X, Y)\xi = 0$, and this proves the claim since also $R^\perp(X, Y)\xi = 0$ from the Ricci equation.

For $\ell = 1$, we have that

$$R^\theta(X, Y)\xi = R_{-\theta}^\ell \left(\alpha_g(X, A_{R_\theta^\ell \xi} Y) - \alpha_g(Y, A_{R_\theta^\ell \xi} X) \right).$$

Since $A_{R_\theta^\ell \xi} = R_\theta A_\xi$, we obtain from the Ricci equation that $R^\theta(X, Y)\xi = R^\perp(X, Y)\xi$ and the claim also follows in this case.

Case 5. Take $\xi \in N_{\ell-1}^g$. Then,

$$\mathcal{R}(X, Y)\xi = B_\theta(X, Y) - B_\theta(Y, X) - B_0(X, Y) + B_0(Y, X)$$

where

$$B_\theta(X, Y) = \pi_\ell(\nabla_X^\perp \pi_{\ell-1}(\nabla_Y^\perp \xi)) + \pi_\ell(\nabla_X^\perp \pi_\ell(\nabla_Y^\perp \xi)) + \pi_{\ell+1}(\nabla_X^\perp R_\theta^\ell \pi_\ell(\nabla_Y^\perp \xi)).$$

It follows that

$$\begin{aligned} \mathcal{R}(X, Y)\xi &= (R_{-\theta}^\ell - I) \left(D_X^\ell(S_Y^{\ell-1}\xi) - D_Y^\ell(S_X^{\ell-1}\xi) + S_X^{\ell-1}D_Y^{\ell-1}\xi - S_Y^{\ell-1}D_X^{\ell-1}\xi - S_{[X, Y]}^{\ell-1}\xi \right) \\ &\quad + S_X^\ell R_\theta^\ell S_Y^{\ell-1}\xi - S_Y^\ell R_\theta^\ell S_X^{\ell-1}\xi, \end{aligned}$$

and the claim follows from (9) and (19).

To conclude the proof, we observe that the case $\tau_g^o = \tau_g - 1$ and $\xi \in N_{\tau_g}^g$ is included in the above cases. ■

Proof of Theorem 2. We show that the triple $(\alpha_g, \langle \cdot, \cdot \rangle, \nabla^\theta)$ satisfies the Gauss, Codazzi and Ricci equations. Then, according to the Fundamental theorem of submanifolds there exists an isometric immersion $g_\theta: L^2 \rightarrow \mathbb{Q}_c^N$ and a parallel vector bundle isometry $\phi_\theta: (N_g L, \nabla^\theta) \rightarrow (Ng_\theta L, \nabla^\perp(g_\theta))$ such that

$$\alpha_{g_\theta}(X, Y) = \phi_\theta(\alpha_g(X, Y))$$

for any $X, Y \in TL$.

The Gauss equation holds since the second fundamental form remains the same. The Codazzi equation

$$(\nabla_X^\theta \alpha_g)(Y, Z) = (\nabla_Y^\theta \alpha_g)(X, Z)$$

is trivially satisfied for $\ell \geq 3$ since $\nabla^\theta \alpha_g = \nabla^\perp \alpha_g$. For $\ell = 1$, we obtain

$$(\nabla_X^\theta \alpha_g)(Y, Z) = \pi_1((\nabla_X^\perp \alpha_g)(Y, Z)) + \pi_2((\nabla_X^\perp \alpha_g)(Y, R_\theta Z))$$

while for $\ell = 2$, we have

$$(\nabla_X^\theta \alpha_g)(Y, Z) = (\pi_1 + \pi_2)((\nabla_X^\perp \alpha_g)(Y, Z))$$

and again the Codazzi equation follows.

The proof that the curvature tensor R^θ of ∇^θ satisfies the Ricci equation

$$R^\theta(X, Y)\xi = \alpha_g(X, A_\xi Y) - \alpha_g(Y, A_\xi X)$$

is a consequence of Lemma 7.

Finally, the statements on the fundamental forms is part of Proposition 8 given below. ■

The following result provides the expressions for the higher fundamental forms of the associated family g_θ in terms of the ones corresponding to g . We observe that they can be used to give an alternative (but more complicated) definition for the associated family by means of the version of the Fundamental theorem of submanifolds coming out from the Burstin-Mayer-Allendoerfer theory.

Proposition 8. *Let $g: L^2 \rightarrow \mathbb{Q}_c^N$ be a simply connected elliptic surface with \mathcal{E}_ℓ^g a circle for some $1 \leq \ell \leq \tau_g^o - 1$. Then, up to identification, the higher fundamental forms of g_θ are given by*

$$\alpha_{g_\theta}^s(X_1, \dots, X_s) = \begin{cases} \alpha_g^s(X_1, \dots, X_s) & \text{if } 2 \leq s \leq \ell + 1, \\ R_\theta^{s-1} \alpha_g^s(X_1, \dots, X_s) = \alpha_g^s(R_\theta X_1, \dots, X_s) & \text{if } \ell + 2 \leq s \leq \tau_g^o + 1. \end{cases}$$

Proof: Since $\alpha_{g_\theta}^2 = \alpha_g$, the case $2 \leq s \leq \ell$ follows easily from the definitions. For the other cases, we have

$$\begin{aligned}\alpha_{g_\theta}^{\ell+1}(X_1, \dots, X_{\ell+1}) &= \pi_\ell(\nabla_{X_{\ell+1}}^\theta \alpha_{g_\theta}^\ell(X_1, \dots, X_\ell)) = \pi_\ell(\nabla_{X_{\ell+1}}^\theta \alpha_g^\ell(X_1, \dots, X_\ell)) \\ &= \pi_\ell(\nabla_{X_{\ell+1}}^\perp \alpha_g^\ell(X_1, \dots, X_\ell)) = \alpha_g^{\ell+1}(X_1, \dots, X_{\ell+1})\end{aligned}$$

and

$$\begin{aligned}\alpha_{g_\theta}^{\ell+2}(X_1, \dots, X_{\ell+2}) &= \pi_{\ell+1}(\nabla_{X_{\ell+2}}^\theta \alpha_{g_\theta}^{\ell+1}(X_1, \dots, X_{\ell+1})) \\ &= \pi_{\ell+1}(\nabla_{X_{\ell+2}}^\theta \alpha_g^{\ell+1}(X_1, \dots, X_{\ell+1})) = \pi_{\ell+1}(\nabla_{X_{\ell+2}}^\perp R_\theta^\ell \alpha_g^{\ell+1}(X_1, \dots, X_{\ell+1})) \\ &= \pi_{\ell+1}(\nabla_{X_{\ell+2}}^\perp \alpha_g^{\ell+1}(R_\theta X_1, \dots, X_{\ell+1})) = \alpha_g^{\ell+2}(R_\theta X_1, \dots, X_{\ell+2}) \\ &= R_\theta^{\ell+1} \alpha_g^{\ell+2}(X_1, \dots, X_{\ell+2}),\end{aligned}$$

and the remaining of the proof is immediate. ■

Proof of Theorem 4. Suppose that the surfaces $g_\theta, g_{\tilde{\theta}} \in G_\ell$ are congruent. Without loss of generality we may assume that $\tilde{\theta} = 0$. Then there exists a parallel vector bundle isometry $\psi: N_g L \rightarrow N_{g_\theta} L$ such that $\psi(N_s^g) = N_s^{g_\theta}$ and

$$\alpha_{g_\theta}^s(X_1, \dots, X_s) = \psi \alpha_g^s(X_1, \dots, X_s)$$

for any $2 \leq s \leq \tau_g^o$. It follows from Proposition 8 that

$$\psi \alpha_g^s(X_1, \dots, X_s) = \begin{cases} \alpha_g^s(X_1, \dots, X_s) & \text{if } 2 \leq s \leq \ell + 1, \\ R_\theta^{s-1} \alpha_g^s(X_1, \dots, X_s) & \text{if } \ell + 2 \leq s \leq \tau_g^o + 1. \end{cases}$$

Therefore,

$$\psi = I \text{ on } N_1^g \oplus \dots \oplus N_\ell^g \text{ and } \psi = R_\theta^s \text{ on } N_s^g \text{ if } s \geq \ell + 1. \quad (22)$$

We conclude that R_θ^s is an isometry for $s \geq \ell + 1$ and hence J_s is an isometry for $s \geq \ell$.

Conversely, suppose that any $\mathcal{E}_s(g)$ is a circle for all $s \geq \ell$, or equivalently, that R_θ^s is an isometry for $s \geq \ell$. Then $\psi: N_g L \rightarrow N_{g_\theta} L$ given by (22) is a vector bundle isometry that preserves the second fundamental form. To conclude the proof it remains to show that ψ is parallel, i.e., $\psi \nabla_X^\perp \xi = \nabla_X^\theta \psi \xi$. For that we distinguish several cases:

Case 1. The case $\xi \in N_1^g \oplus \dots \oplus N_{\ell-1}^g$ is trivial.

Case 2. Assume that $\xi \in N_\ell^g$. We have using (5) that

$$\begin{aligned}\nabla_X^\theta \psi \xi &= \nabla_X^\theta \xi = (\pi_{\ell-1} + \pi_\ell)(\nabla_X^\perp \xi) + \pi_{\ell+1}(\nabla_X^\perp R_\theta^\ell \xi) \\ &= (\pi_{\ell-1} + \pi_\ell)(\nabla_X^\perp \xi) + R_\theta^{\ell+1} \pi_{\ell+1}(\nabla_X^\perp \xi) = \psi \nabla_X^\perp \xi.\end{aligned}$$

Case 3. Assume that $\xi \in N_{\ell+1}^g$. We have,

$$\nabla_X^\theta \psi \xi = \nabla_X^\theta R_\theta^{\ell+1} \xi = R_{-\theta}^\ell \pi_\ell (\nabla_X^\perp R_\theta^{\ell+1} \xi) + (\pi_{\ell+1} + \pi_{\ell+2}) (\nabla_X^\perp R_\theta^{\ell+1} \xi)$$

and

$$\psi \nabla_X^\perp \xi = \pi_\ell (\nabla_X^\perp \xi) + R_\theta^{\ell+1} \pi_{\ell+1} (\nabla_X^\perp \xi) + R_\theta^{\ell+2} \pi_{\ell+2} (\nabla_X^\perp \xi).$$

To obtain equality we observe that (6) yields

$$R_{-\theta}^\ell \pi_\ell (\nabla_X^\perp R_\theta^{\ell+1} \xi) = \pi_\ell (\nabla_X^\perp \xi)$$

and that the $N_{\ell+1}^g$ and $N_{\ell+2}^g$ components are equal due to (12) and (5), respectively.

Case 4. Assume that $\xi \in N_s^g$ for $s \geq \ell + 2$. We have,

$$\nabla_X^\theta \psi \xi = \nabla_X^\perp R_\theta^s \xi = (\pi_{s-1} + \pi_s + \pi_{s+1}) (\nabla_X^\perp R_\theta^s \xi),$$

$$\psi \nabla_X^\perp \xi = R_\theta^{s-1} \pi_{s-1} (\nabla_X^\perp \xi) + R_\theta^s \pi_s (\nabla_X^\perp \xi) + R_\theta^{s+1} \pi_{s+1} (\nabla_X^\perp \xi)$$

and equality follows from (5), (6) and (12). ■

Proof of Corollary 5. The proof follows from Theorem 4 and the fact obtained in [8] that an elliptic surface has circular curvature ellipses from some order on if and only if any polar surface has circular curvature ellipses up to that order. ■

To conclude this section, we briefly discuss the situation in which more than one ellipse of curvature of an elliptic surface is a circle.

If two ellipses \mathcal{E}_r^g and \mathcal{E}_t^g of $g: L^2 \rightarrow \mathbb{Q}_c^N$ are circles, then the *extended associated family* $G_{r,t}$, $r < t$, is the set of isometric immersions obtained by taking the associated families of all the elements of G_r with respect to \mathcal{E}_t^g or, equivalently, of G_t with respect to \mathcal{E}_r^g . Since this definition can be extended to any number of circular ellipses of curvature, we denote by G_{r_1, \dots, r_k} , $r_1 < \dots < r_k$, the extended associated family corresponding to the circular ellipses $\mathcal{E}_{r_1}^g, \dots, \mathcal{E}_{r_k}^g$.

Assume that two consecutive ellipses \mathcal{E}_ℓ^g and $\mathcal{E}_{\ell+1}^g$ are circles but that \mathcal{E}_k^g is not a circle for some $k \geq \ell + 2$. We claim that in this case the extended associated family satisfies $G_{\ell, \ell+1} = G_{\ell+1}$. To prove the claim, take $g_{(\theta_1, \gamma_1)}^1, g_{(\theta_2, \gamma_2)}^2 \in G_{\ell, \ell+1}$. If ∇^j denotes the connection in $N_{g^j} L$, $j = 1, 2$, we have from (13) that

$$\begin{cases} \nabla_X^j \xi = (\pi_{\ell-1} + \pi_\ell) (\nabla_X^\perp \xi) + \pi_{\ell+1} (\nabla_X^\perp R_{\theta_j}^\ell \xi) & \text{if } \xi \in N_\ell^g \\ \nabla_X^j \eta = R_{-\theta_j}^\ell \pi_\ell (\nabla_X^\perp \eta) + \pi_{\ell+1} (\nabla_X^\perp \eta) + \pi_{\ell+2} (\nabla_X^\perp R_{\theta_j}^{\ell+1} \eta) & \text{if } \eta \in N_{\ell+1}^g \\ \nabla_X^j \delta = R_{-\theta_j}^{\ell+1} \pi_{\ell+1} (\nabla_X^\perp \delta) + (\pi_{\ell+2} + \pi_{\ell+3}) (\nabla_X^\perp \delta) & \text{if } \delta \in N_{\ell+2}^g. \end{cases} \quad (23)$$

Now suppose that $g_{(\theta_1, \gamma_1)}^1$ and $g_{(\theta_2, \gamma_2)}^2$ are congruent. Then, there is a parallel vector bundle isometry $T: N_{g^1} L \rightarrow N_{g^2} L$ such that $T(N_{g^1}^s) = N_{g^2}^s$ and

$$\alpha_{g^2}^s(X_1, \dots, X_s) = T \alpha_{g^1}^s(X_1, \dots, X_s)$$

for any $2 \leq s \leq \tau_g^o$. It follows from Proposition 8 that

$$\alpha_{g^j}^s(X_1, \dots, X_s) = \begin{cases} \alpha_g^s(X_1, \dots, X_s) & \text{if } 2 \leq s \leq \ell + 1, \\ R_{\theta_j}^{s-1} \alpha_g^s(X_1, \dots, X_s) & \text{if } s = \ell + 2, \\ R_{\theta_j + \gamma_j}^{s-1} \alpha_g^s(X_1, \dots, X_s) & \text{if } \ell + 3 \leq s \leq \tau_g^o + 1. \end{cases}$$

Hence, we have $T = I$ on $N_1^g \oplus \dots \oplus N_\ell^g$,

$$T \circ R_{\theta_1}^{\ell+1} = R_{\theta_2}^{\ell+1} \text{ on } N_{\ell+1}^g, \text{ and } T \circ R_{\theta_1 + \gamma_1}^s = R_{\theta_2 + \gamma_2}^s \text{ on } N_s^g \text{ if } s \geq \ell + 2.$$

Thus $T = R_{\theta_2 - \theta_1}^{\ell+1}$ on $N_{\ell+1}^g$. Moreover, since not all ellipse are circles we must have that

$$\theta_1 + \gamma_1 \equiv \theta_2 + \gamma_2 \pmod{\pi} \quad (24)$$

and $T = I$ on N_s^g if $s \geq \ell + 2$.

A straightforward computation using (23) shows that if (24) holds, then the vector bundle isometry $T: N_{g^1}L \rightarrow N_{g^2}L$ defined as $T = R_{\theta_2 - \theta_1}^{\ell+1}$ on $N_{\ell+1}^g$ and as the identity otherwise is parallel, that is, $T \nabla_X^1 \mu = \nabla_X^2 T \mu$ for any $\mu \in N_{g^1}L$. This concludes the proof of the claim.

We have proved the following result that, together with our other results, has to be used for the proof of Theorem 1.

Proposition 9. *Let $g: L^2 \rightarrow \mathbb{Q}_c^N$ be a simply connected elliptic surface such that the ellipses $\mathcal{E}_j^g, \ell \leq j \leq \ell + r$, are all circles. Then, that $G_{\ell, \dots, \ell+r} = G_{\ell+r}$ holds.*

6 Minimal submanifolds of rank 2

Let $f: M^n \rightarrow \mathbb{R}^N$, $n \geq 3$, be a submanifold of rank two. This means that the relative nullity subspaces $\Delta(x) \subset T_x M$ defined by

$$\Delta(x) = \{X \in T_x M : \alpha_f(X, Y) = 0 \text{ for all } Y \in T_x M\}$$

form a codimension two subbundle of the tangent bundle. The submanifold is called *elliptic* if there exists an almost complex structure $J: \Delta^\perp \rightarrow \Delta^\perp$ such that

$$\alpha_f(X, X) + \alpha_f(JX, JX) = 0 \text{ for all } X \in \Delta^\perp.$$

Hence f is minimal if and only if J is orthogonal. As in the case of elliptic surfaces, the normal bundle splits as

$$N_f M = N_1^f \oplus \dots \oplus N_{\tau_f}^f.$$

It was shown in [8] that everything explained in this paper about polar surfaces to elliptic surfaces extends to this case. In particular, any elliptic submanifold in case (ii) admits locally many polar surfaces which turn out to be elliptic surfaces.

Hereafter, we assume that $f: M^n \rightarrow \mathbb{R}^N$ is minimal and simply connected of rank two. For any $\varphi \in \mathbb{S}^1 = [0, \pi)$ consider the tensor field R_φ that is the identity on Δ and the rotation through φ in Δ^\perp . It was observed in [9] that the normal valued tensor field given by

$$\alpha_\varphi(X, Y) = \alpha_f(R_\varphi X, Y),$$

satisfies the Gauss, Codazzi and Ricci equations with respect to the normal connection of f . Hence, for each $\varphi \in \mathbb{S}^1$ there exists a minimal submanifold $f_\varphi: M^n \rightarrow \mathbb{R}^N$ of rank two that forms the associated family of f .

According to the polar parametrization given in [8, Thm. 10], minimal submanifolds of rank two can be described parametrically along a subbundle of the normal bundle of an elliptic surface whose curvature ellipse of a specific order is circular. More precisely, given an elliptic surface $g: L^2 \rightarrow \mathbb{Q}_c^{N-c}$, $c = 0, 1$, with \mathcal{E}_ℓ^g for some $1 \leq \ell \leq \tau_g^o - 1$ a circle, the map $f: \Lambda_\ell \rightarrow \mathbb{R}^N$ defined by

$$f(\delta) = h(x) + \delta, \quad \delta \in \Lambda_\ell(x),$$

where $\Lambda_\ell = N_{\ell+1}^g \oplus \cdots \oplus N_{\tau_g}^g$ and h is any ℓ -cross section to g , is at regular points a minimal submanifold of rank two with polar surface g . Conversely, any minimal submanifold of rank two admits locally such a parametrization with g a polar map.

The recursive procedure for the construction of the cross sections [8, Prop. 6] yields

$$h = c\omega g + \text{grad } \omega + \gamma_0 + \gamma_1 + \cdots + \gamma_\ell,$$

where ω is a solution of the linear elliptic differential equation

$$\Delta u + \langle X, \text{grad } u \rangle + c\lambda u = 0$$

for suitable $X \in TL$ and $\lambda \in C^\infty(L)$, γ_0 is any section in Λ_ℓ , $\gamma_1 \in N_1^g$ is the unique solution of $A_{\gamma_1} = \text{Hess}_\omega + c\omega I$ and $\gamma_j \in N_j^g$, $2 \leq j \leq \ell$, where L^2 is endowed with the metric which makes J orthogonal.

Take $g_\theta \in G_\ell$ and the corresponding vector bundle isometry $\phi_\theta: N_g L \rightarrow N_{g_\theta} L$. Then,

$$h_\theta = c\omega g_\theta + \text{grad } \omega + \phi_\theta \gamma_0 + \phi_\theta \gamma_1 + \cdots + \phi_\theta \gamma_\ell$$

is an ℓ -cross section to g_θ . With these elements we have the following result.

Theorem 10. *Any submanifold in the associated family of a minimal submanifold $f: M^n \rightarrow \mathbb{R}^N$ of rank two can be locally parametrized as*

$$f_\theta(\delta) = h_\theta(x) + \phi_\theta \delta, \quad \delta \in \Lambda_\ell(x).$$

Proof: It is easy to check that f_θ is isometric to f , has the same normal connection and its second fundamental form is given by

$$\alpha_{f_\theta}(X, Y) = \alpha_f(R_{-\theta}X, Y).$$

In particular, f_θ is minimal and thus belongs to the associated family of f . ■

The above discussion allows us to give an answer to the question of which minimal submanifolds of rank two have trivial associated family. This is the case if and only if the associated family of its polar surfaces is trivial, which is equivalent to the fact that a (local) bipolar surface to f , i.e., any polar surface to its polar surface, is m -isotropic.

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